Lecture 9 - The Curious Case of Discontinuities

A Puzzle...

Electron Jelly

Imagine a sphere of radius a filled with negative charge of uniform density, the total charge being equivalent to that of two electrons. Assume that two protons are embedded in this jelly, and further suppose that, in spite of their presence, the negative charge distribution remains uniform. Where must the protons be located so that the force on each of them is zero?
(This is a surprisingly realistic caricature of a hydrogen molecule; the magic that keeps the electron cloud in the molecule from collapsing around the protons is explained by quantum mechanics!)

<u>Solution</u>
Recalling Gauss's Law, the electric field at a distance $r \leq a$ from the center of the jelly sphere (which has charge density $\rho = -\frac{2e}{\frac{4}{3}\pi a^3}$) equals $E\left(4\pi r^2\right) = \frac{1}{\epsilon_0}\left(\frac{4}{3}\pi r^3 \rho\right)$ or equivalently

$$E = \frac{r\rho}{3\epsilon_0} \qquad (1)$$

A proton at a distance r would feel the force $F_{\text{jelly}} = E\,e = \frac{r|\rho|e}{3\epsilon_0}$ pulling it towards the center of the jelly. To have stable equilibrium, the other proton must also be at a distance r on the same diameter as the first proton, but in the opposite direction. The repelling force between the two protons equals $F_{\text{proton}} = \frac{k e^2}{(2r)^2}$. Equilibrium occurs when $F_{\text{jelly}} = F_{\text{proton}}$, which allows us to solve for r,

$$\begin{aligned} \frac{r|\rho|e}{3\epsilon_0} &= \frac{k e^2}{(2r)^2} \\ \frac{r e}{3\epsilon_0}\frac{2e}{\frac{4}{3}\pi a^3} &= \frac{k e^2}{(2r)^2} \\ \frac{r^3}{a^3} &= \frac{1}{8} \\ r &= \frac{a}{2} \end{aligned} \qquad (2)$$

In retrospect, this factor of $\frac{1}{2}$ is clear. If all of the $-2e$ electron charge were located in a point charge at the center, it would provide a force on one of the protons that is 8 times the force due to the other proton (because the other proton is twice as far away, and half as big). So the forces will balance if we reduce the effective electron charge by a factor of 8. This is accomplished by reducing the effective radius of the jelly by a factor of 2. □

Advanced Electrostatics Problems

An Infinite Sheet

<u>Example</u>

An infinite sheet with uniform charge density σ lies in the x-y plane. What is the electric field $\vec{E}$ at all points in space?

Solution

We have built up a lot of mathematical machinery, so let's solve this problem in four different ways:

- Gauss's Law: Exploit symmetry and use $\int \vec{E}\cdot d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$ to determine the electric field of the sheet
- Coulomb's Law: Consider the sheet to be comprised of uniformly charged rods, whose electric field we found in the last lecture is $\vec{E}_{\text{rod}} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$. Using Coulomb's Law and the Principle of Superposition ($\vec{E} = \int \frac{k\,dq}{r^2}\hat{r}$), add up the electric fields of all the rods to determine the electric field from the sheet
- Coulomb's Law: The Principle of Superposition says that we can break up our charge distribution however we want, which grants us a lot of flexibility. Consider the sheet to be comprised of small patches of area ($dx\,dy$) in Cartesian coordinates and sum up the electric field from each patch
- Coulomb's Law: Cartesian coordinates are sometimes not the optimal coordinate system. Consider the sheet to be comprised of small patches of area ($r\,dr\,d\theta$) in polar coordinates and sum up the electric field from each patch

Method 1

By reflection symmetry, the electric field from the sheet must point in the z-direction, and by translational symmetry it can only depend upon the distance z from the sheet and not on the Cartesian coordinates x or y. In other words, $\vec{E} = E[z]\,\hat{z}$ for $z > 0$, and by symmetry $\vec{E} = -E[z]\,\hat{z}$ for $z < 0$.

We can exploit this symmetry by considering an (imaginary) cube with side length $2s$ that is split in two by the plane. As shown below, $d\hat{a}$ for this cube points in the x- or y-directions for four of the cube's faces (the ones that get intersecting the x-y plane), whereas $d\hat{a} = \hat{z}$ for the top face and $d\hat{a} = -\hat{z}$ for the bottom face.

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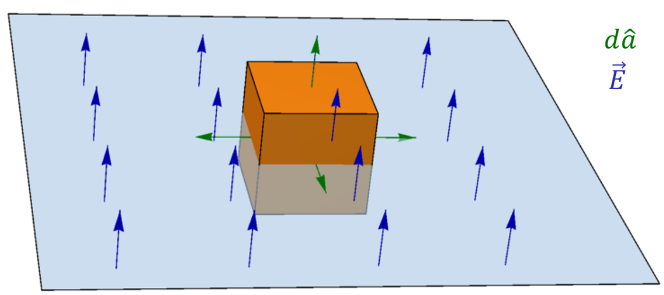

Since $\vec{E}$ points in the z-direction, the surface integral $\int \vec{E}\cdot d\vec{a}$ will be zero over the four faces intersecting the x-y plane. For the top face of the cube, $\int \vec{E}\cdot d\vec{a} = \int (E[z]\,\hat{z})\cdot(\hat{z}\,d\vec{a}) = \int E[z]\,da$. Similarly, for the bottom face $\int \vec{E}\cdot d\vec{a} = \int (-E[z]\,\hat{z})\cdot(-\hat{z}\,d\vec{a}) = \int E[z]\,da$. Therefore, Gauss's Law yields

$$\frac{Q_{\text{enc}}}{\epsilon_0} = \int \vec{E}\cdot d\vec{a} = 2\int_{\text{top face}} E[z]\,da = 2\,E[z]\int_{\text{top face}} da = 2\,E[z]\,(2s)^2 \tag{3}$$

where we have used the fact that the cube has side length $2s$ and hence area $(2s)^2$ on each face. The total charge Q_{enc} enclosed by the cube equals the charge of a square of side length $2s$ on the x-y plane, which is $(2s)^2\,\sigma$. Substituting this above, we find

$$\frac{(2s)^2\,\sigma}{\epsilon_0} = 2\,E[z]\,(2s)^2 \tag{4}$$

$$E[z] = \frac{\sigma}{2\,\epsilon_0} \tag{5}$$

By exploiting the directionality of $\vec{E}$ discussed above and using symmetry, we find that the electric field at all points in space equals

$$\vec{E} = \begin{cases} \frac{\sigma}{2\epsilon_0}\hat{z} & z > 0 \\ \vec{0} & z = 0 \\ \frac{\sigma}{2\epsilon_0}(-\hat{z}) & z < 0 \end{cases} \tag{6}$$

This is a remarkable result, and we will examine its implications below, but for now, let us verify this answer by computing the electric field in other ways.

Method 2

If we split up the infinite sheet into thin rods of thickness dx running parallel to the y-axis, their charge density would equal $\lambda = \sigma\, dx$. The contribution of the electric field at our point in the z-direction would equal $dE = \frac{\lambda}{2\pi r \epsilon_0}\frac{z}{r}$ where the last term picks out the z-component. Using $r^2 = x^2 + z^2$, the total electric field equals

$$E = \int_{-\infty}^{\infty} \frac{\sigma\, dx}{2\pi\epsilon_0}\frac{z}{x^2+z^2} = \left(\frac{\sigma}{2\pi\epsilon_0}\text{ArcTan}\left[\frac{x}{z}\right]\right)_{x=-\infty}^{x=\infty} = \frac{\sigma}{2\epsilon_0} \tag{7}$$

as found above.

Method 3

We can always go back to good-old Coulombs Law! Consider a small patch of the surface at a point $(x, y, 0)$ with area $dx\, dy$. The magnitude of the electric field at $(0, 0, z)$ from this patch equals $dE = \frac{k\, dq}{r^2} = \frac{k(\sigma\, dx\, dy)}{x^2+y^2+z^2}$. By symmetry, we know that the electric field must point in the z-direction, and hence we only want to pick out its z-component, which is $\frac{k(\sigma\, dx\, dy)}{x^2+y^2+z^2}\frac{z}{\left(x^2+y^2+z^2\right)^{1/2}}$. Integrating over all patches is nasty, but yields the correct solution

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{k(\sigma\, dx\, dy)}{x^2+y^2+z^2}\frac{z}{\left(x^2+y^2+z^2\right)^{1/2}} = \int_{-\infty}^{\infty}\frac{2k\sigma z\, dy}{y^2+z^2} = 2\pi k\sigma = \frac{\sigma}{2\epsilon_0} \tag{8}$$

Method 4

Similar to Method 3, we now integrate over the x-y plane in polar coordinates. The magnitude of the electric field at $(0, 0, z)$ from a small patch at a point (r, θ) with area $r\, dr\, d\theta$ will be $dE = \frac{k\, dq}{r^2+z^2} = \frac{k(\sigma\, r\, dr\, d\theta)}{r^2+z^2}$. By symmetry, we know that the electric field must point in the z-direction, and hence we only want to pick out its z-component, which is $\frac{k(\sigma\, r\, dr\, d\theta)}{r^2+z^2}\frac{z}{\left(r^2+z^2\right)^{1/2}}$. Integrating over all patches is much nicer in polar coordinates (showing that it often helps to exploit symmetry and work in the most convenient coordinate system),

$$E = \int_0^{2\pi}\int_0^{\infty}\frac{k(\sigma\, r\, dr\, d\theta)}{r^2+z^2}\frac{z}{\left(r^2+z^2\right)^{1/2}} = 2\pi k\sigma = \frac{\sigma}{2\epsilon_0} \tag{9}$$

This quadruply confirms our result! □

Let’s take a second to admire this astounding result! No matter how far away you are from the shell, the electric field has the value $\frac{\sigma}{2\epsilon_0}$ pointing away from the plane. Additionally, no matter how close you come to the plane, you still feel the electric field $\frac{\sigma}{2\epsilon_0}$ pointing away from the plane. However, if you pass through the plane, the electric field will *discontinuously* change from $\frac{\sigma}{2\epsilon_0}$ pointing in one direction to $\frac{\sigma}{2\epsilon_0}$ pointing in the *opposite* direction! Furthermore, if you are a piece of charge embedded within the sheet, you will feel an electric field of 0 (by symmetry), but if you move an infinitesimal distance off the sheet you will feel an electric field of $\frac{\sigma}{2\epsilon_0}$.

This type of behavior is only exhibited by infinitely thin sheets of charge. Volume charge distributions create continuous electric fields (since the integration in $\vec{E} = \int \frac{k\rho\, dx'\, dy'\, dz'}{r^2}\hat{r}$ smooths out discontinuities in ρ). And lines of charges don’t seem as alarming because the electric field goes to ∞ near the line of charge (just like for a point charge). So only 2D charge configurations have this bizarre behavior where the electric field is discontinuous but stays finite everywhere.

As stated in the "Field from a Cylindrical Shell, Right and Wrong" section in the previous lecture, a great rule to remember about discontinuities in nature is:

With all discontinuities in nature, the actual value at the discontinuity equals the *average* of the discontinuous values! (10)

This statement is worth its weight in gold; print it on bumper stickers and share it with the world! You will see it come up again and again in physics (from charge distributions) and mathematics (Fourier series).

Intersecting Sheets

Hole in a Shell

Example

Consider a spherical shell of charge, of radius R and surface charge density σ, from which a small circular piece of radius $b \ll R$ has been removed.

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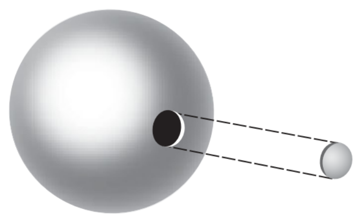

We know that for a complete spherical sheet (the $b = 0$ limit), there will be a discontinuity in the electric field when we go from just inside to just outside the sphere. Once we remove this infinitesimal cap from the spherical shell, will there still be a discontinuity in the electric field? Verify your result.

Solution

Method 1: A clever application of the Principle of Superposition

Imagine that we have a complete spherical shell overlaid with a thin spherical cap of charge density $-\sigma$. By the Principle of Superposition, this is identical to the above setup. Since we can much more easily determine the electric field everywhere for a spherical shell, and at the end of Lecture 6 we determined the electric force (and hence the electric field) on the axis of symmetry from a circular sheet of charge (see the *Force from a Disk* example).

For the spherical shell, Gauss's Law easily yields the electric field $E = 0$ inside the sphere and $E = \frac{\sigma}{\epsilon_0}$ pointing radially outwards. Now we add the effect of the thin hole, which at the midpoint looks like an infinite plane of charge with an electric field $\frac{\sigma}{2\epsilon_0}$ pointing *towards* the plane (since the charge density is $-\sigma$). Thus, inside the shell the electric field will be $0 + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{2\epsilon_0}$ pointing outwards and outside the sphere the electric field will be $\frac{\sigma}{\epsilon_0} - \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{2\epsilon_0}$ pointing outwards. Hence, the electric field is *continuous* with magnitude $\frac{\sigma}{2\epsilon_0}$ across the missing spherical cap.

This is an incredible result! By removing an infinitesimal piece of the sheet, we removed the finite discontinuity that would have existed if $b = 0$. Similarly, if an infinite sheet lies on the x-y plane, and you poke a tiny hole through it, the electric field will be continuous across this hole.

Method 2: Coulomb's Law and the Principle of Superposition

We could double check our solution at the point right in the middle of the spherical cap using straight-up integration. Instead of integrating over the entire sphere. Note that if the aperture has radius b, then θ spans from $\frac{b}{R}$ to π.

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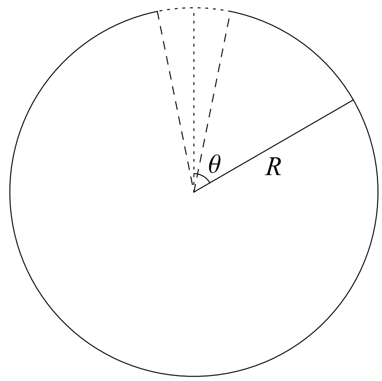

Note that the distance from the point at angle θ to the middle of the aperture equals $2\,R\,\text{Sin}\left[\frac{\theta}{2}\right]$ (which can either be found through the Law of Cosines or simple geometry). Therefore, the electric field in the middle of the aperture (which only has a component in the vertical direction) has magnitude

$$\begin{aligned} E &= \int_0^{2\pi}\int_{b/R}^{\pi} \frac{k\left(R^2\,\text{Sin}[\theta]\,\sigma\right)}{4\,R^2\,\text{Sin}\left[\frac{\theta}{2}\right]^2}\,\text{Sin}\left[\frac{\theta}{2}\right] d\theta\, d\phi \\ &= \frac{\pi\,k\,\sigma}{2}\int_{b/R}^{\pi}\frac{\text{Sin}[\theta]}{\text{Sin}\left[\frac{\theta}{2}\right]}\,d\theta \\ &= \pi\,k\,\sigma\int_{b/R}^{\pi}\text{Cos}\left[\frac{\theta}{2}\right] d\theta \\ &= 2\,\pi\,k\,\sigma\left(\text{Sin}\left[\frac{\theta}{2}\right]\right)_{\theta=b/R}^{\theta=\pi} \\ &= 2\,\pi\,k\,\sigma\left(1-\text{Sin}\left[\frac{b}{2\,R}\right]\right) \end{aligned} \tag{13}$$

In the limit $b \ll R$, $\text{Sin}\left[\frac{b}{2\,R}\right] \approx \frac{b}{2\,R} \approx 0$ so that $E \approx 2\,\pi\,k\,\sigma = \frac{\sigma}{2\,\epsilon_0}$, as desired. You are highly encouraged to modify this calculation to compute the electric field at a distance $r = R - \epsilon$ and $r = R + \epsilon$ from the origin along the line of symmetry of the spherical cap to double check that the electric field is indeed continuous. □

A Plane and a Slab

Advanced Section: Maximum Field from a Blob

Advanced Section: Ball in a Sphere

Recommended Problems

This is a list of excellent problems (with solutions) in David Morin's book.

- 1.6 Zero potential energy for equilibrium

Mathematica Initialization